The Arrow Polynomial of Periodic Virtual Links

Sarah Dorich

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Abstract

Virtual knot theory is a generalization of classical knot theory. A periodic link is one with rotational symmetry. Murasugi (1988) proved a simple relationship between the Jones polynomial of a periodic link and the Jones polynomial of its factor link. In virtual knot theory, the arrow polynomial is an analog of the Jones polynomial. This work extends Murasugi's relationship to the arrow polynomial for virtual periodic links.

1 Virtual knot theory

Louis Kauffman introduced virtual knot theory in 1999. Classical knot theory studies the embeddings of curves in S^3. By removing two points from S^3, we obtain $S^2 \times I$. Therefore, classical knot theory can be regarded as the study of embeddings of circles in $S^2 \times I$, i.e. the thickened two-sphere. Virtual knot theory is a natural extension of classical knot theory. Virtual knot theory studies the embeddings of curves in any thickened orientable surfaces of arbitrary genus. Note that surfaces in which virtual knots are embedded needs to be thickened because of classical crossings. On the right of Figure 1, a virtual knot is embedded on the surface of a thickened torus. The dotted line represents the part of the knot that is on the "bottom" of the torus while the solid line represents the part of the knot that is on the "top" of the torus. As Figure 1 illustrates, when projected onto $\mathbb{R}^2$, an intersection between the dotted and solid line is created and may

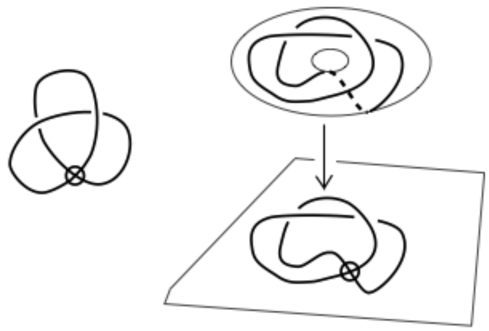

Figure 1: Surfaces and virtuals

be thought of as an artificial crossing. This crossing is called a virtual crossing. The picture on the left of Figure 1 is the corresponding virtual knot diagram. A virtual crossing is neither a over nor under crossing and is illustrated diagramatically by placing a small circle around the crossing point. The virtual knot depicted in Figure 1 has one virtual crossing and two classical crossings. For a more complete introduction to virtual knot theory see [ref].

2 The arrow polynomial

An important invariant for oriented virtual links is called the *arrow polynomial*. It is a natural extension of the Jones polynomial (Kauffman bracket version) but is a considerably stronger invariant due to the extra structure in virtual links.

Let D be a diagram of an oriented virtual link L. We can create states by resolving crossings in the standard way [ref (modern knot theory text)]. An oriented state expansion means that each local resolution is either an oriented smoothing or a *disoriented* resolution. As illustrated in Figure 2, a disoriented resolution is one where the resolution results in a discontinuity which we designate with a cusp. If a resolution is disoriented, a cusp will form on each side of the splitting marker implying that cusps always come in pairs. For this reason, we refer to this extra structure in virtual states as cusp pairs. In the Kauffman bracket polynomial, we ignore orientation when calculating the state expansion of a classical knot. If we didn't ignore orientation before resolving crossings in a classical knot, we would not obtain any extra information because cusp pairs would always “cancel” with each other according to a set of cancellation rules described later. However, by not ignoring orientation when calculating the state expansion for a virtual link, virtual crossings can cause cusp pairs not to cancel. We use non-cancelling cusp pairs that arise in virtual states to add extra variables in the arrow polynomial. This is the reason that the arrow polynomial is a stronger invariant for virtual links than the Jones polynomial is for classical links.

Figure 2: Resolutions of a classical crossing

Let S be a state of the diagram D and let $p(S)$ and $n(S)$ be, respectively, the number of splitting markers in S joining positive and negative regions. Then define $\langle D \mid S \rangle = A^{p(S)-n(S)}$ and $|S|$ to be the number of components in the state S.

The normalized arrow polynomial for L is given by the formula:

$$\mathcal{A}\left[L\right] = \left(-A^3\right)^{-w(D)} \sum_{S} \langle D \mid S \rangle \, d^{|S|-1} \left[S\right] \tag{1}$$

where S runs over the oriented bracket states of the diagram, $d = -A^2 - A^{-2}$, $w(D)$ is the writhe of the virtual knot diagram, and $[S]$ is a product of extra variables, K_1, K_2, $\ldots$, K_n associated with the state S. These variables are explained below.

Each disoriented resolution gives rise to a cusp pair where each cusp has either two oriented strands going into the cusp or two oriented strands leaving the cusp. We reduce this structure according to a set of rules that yields invariance under the virtual Reideimeister moves [ref]. The basic conventions for this simplification are shown in Figure 3. The angle locally divides the plane into two parts: one part is the span of the acute angle and the other part is the span of the obtuse angle. We refer to the span of the acute angle as the *inside* of the cusp. A cusp pair cancels if the inside of two adjacent cusps are on the same side of the strand that connects them. A cusp pair does not cancel if the inside of one cusp is on one side of the strand that connects them while the inside of another cusp is on the outside of that strand. Note that non-cancellation may only happen if there are an odd number of virtual crossings in between the cusp pairs. This explains why cusps in states of a classical link always cancel and thus an oriented state expansion would be excessive in this case. Say an

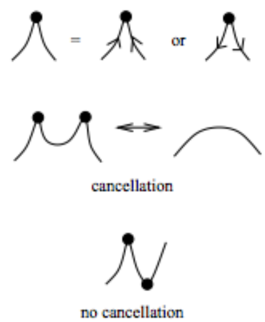

Figure 3: Reduction of oriented states via the arrow convention

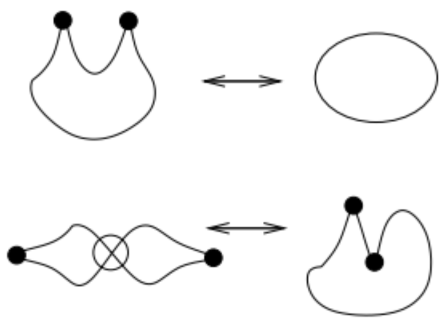

Figure 4: Examples of reduction of oriented states

arbitrary state S has a single component with n non-cancelling cusp pairs. Then that component will contribute K_n to $\mathcal{A}$. If S has, say, m components that each have n non-cancelling cusp pairs then S will contribute K_n^m to $\mathcal{A}$. For example, consider Figure 5, the virtualized trefoil and Figure 6 which shows its states. Starting from the top left, the first state shown in Figure 6 contains no cusps implying that the extra variable associated with this state is just 1. The next state to the right contains one cusp pair; however, this cusp pair cancels and the extra variable associated with this state is just 1 again. The same is true for the third, fourth and fifth states. The sixth, seventh, and eighth states are more interesting. For example, consider the eighth state (the bottom right state in Figure 6). It contains 3 components. The outside component yields one non-cancelling cusp pair and thus contributes a K_1 to the arrow polynomial. The top inner component

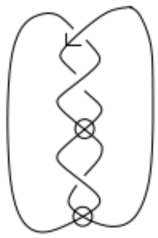

Figure 5: Virtualized trefoil

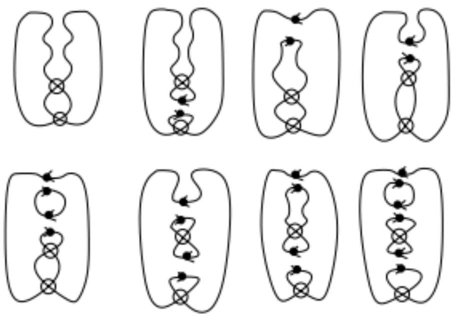

Figure 6: States of the virtualized trefoil

has a cusp pair which cancels and the bottom inner component yields one non-cancelling cusp pair (which contributes K_1). Thus, the extra variable associated with this state is K_1^2. States six and seven also happen to contribute K_1^2.

3 Periodic virtual links

Let L be an oriented virtual link embedded in $\Sigma \times I$ where Σ is a surface of arbitrary genus. Suppose that there exists an orientation-preserving auto-homeomorphism $\phi : \Sigma \times I \to \Sigma \times I$ of order r which maps L onto itself. Then L is a periodic virtual link of order r. Let $\Sigma' \times I = (\Sigma \times I)/\phi$ be the quotient space under ϕ and let $\psi : \Sigma \to \Sigma'$ be the covering projection. Let $L_* = \psi(L)$. L_* is called the virtual factor link.

Now let's consider the link diagram $\tilde{L}$ of L. Let ζ be the rotation of $\mathbb{R}^2$ about the origin 0 through $2\pi/r$. Let $R(0, 2\pi/r)$ be the closed domain bounded by two half lines $\theta = 0$ and $\theta = 2\pi/r$. Define $\tilde{L}_0 = \tilde{L} \cap R(0, 2\pi/r)$. Since L is a virtual link having period r, L has a diagram $\tilde{L}$ in $\mathbb{R}^2$ which is divided into r pieces, $\tilde{L}_0, \tilde{L}_1, \ldots, \tilde{L}_{r-1}$ such that $\zeta\left(\tilde{L}_i\right) = \tilde{L}_{i+1}$, $0 \leq i \leq r-1$ and $\tilde{L}_0 = \tilde{L}_r$. Let $A_1, A_2, \ldots, A_l$ be the points of intersection of $\tilde{L}_0$ and the line $\theta = 0$. Let $\zeta(A_i) = B_i$, $0 \leq i \leq l$. Then by joining A_i and B_i in $\mathbb{R}^2$ by an arc starting at B_i on the half line $\theta = 2\pi/r$ and ending at A_i, we obtain a diagram $\tilde{L}_*$ of the factor link $L_* = \psi(L)$.

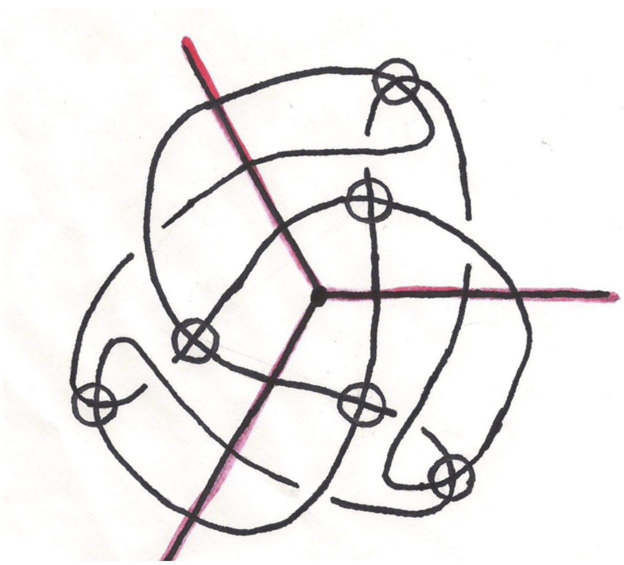

Figure 7: $\tilde{L}$

We now establish a preliminary result.

Lemma 1. *Let*

$$\xi_r(A) = \left(\sum_{j=0}^{r-1} (-1)^j A^{-4j}\right) - A^{2-2r}.$$

Then for prime r and for all $k \in \mathbb{N}$,

$$d^{(r-1)(k-1)} \equiv 1 \pmod{r, \xi_r(A)}.$$

Proof. If $k = 1$, the exponent is zero so the statement is obvious.

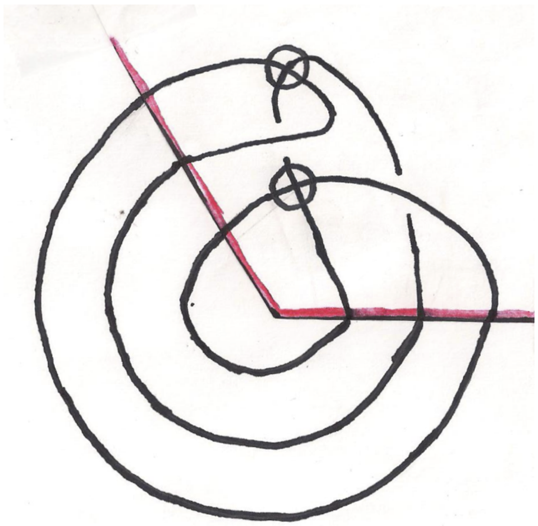

Figure 8: $\tilde{L}_*$

For $k \geq 2$,

$$\begin{aligned} d^{(r-1)(k-1)} - 1 &= \left(d^{r-1}\right)^{k-1} - 1 \\ &= \left(d^{r-1} - 1\right)\left(\left(d^{r-1}\right)^{k-2} + \cdots + 1\right). \end{aligned}$$

If we show that

$$d^{r-1} - 1 \equiv 0 \pmod{r, \xi_r(A)},$$

we are finished. Recall from Section 2 that $d = -A^2 - A^{-2}$. Using the binomial theorem we obtain

$$\begin{aligned}
d^{r-1}-1 &= \left[\left(-A^2-A^{-2}\right)^{r-1}\right]-1 \\
&= \left[(-1)^{r-1}\left(A^2+A^{-2}\right)^{r-1}\right]-1 \\
&= \left[(-1)^{r-1}\sum_{j=0}^{r-1}\binom{r-1}{j}A^{2(r-j-1)}A^{-2j}\right]-1 \\
&= \left[(-1)^{r-1}\sum_{j=0}^{r-1}\binom{r-1}{j}A^{2r-2}A^{-4j}\right]-1 \\
&= \left[(-1)^{r-1}A^{2r-2}\sum_{j=0}^{r-1}\binom{r-1}{j}A^{-4j}\right]-1.
\end{aligned}$$

If $r=2$,

$$(-1)^{r-1}\equiv 1 \pmod 2.$$

If $r\neq 2$, it must be an odd prime. Then $r-1$ is even and $(-1)^{r-1}=1$. Therefore,

$$\begin{aligned}
&\left[(-1)^{r-1}A^{2r-2}\sum_{j=0}^{r-1}\binom{r-1}{j}A^{-4j}\right]-1 \\
&\equiv \left[A^{2r-2}\sum_{j=0}^{r-1}\binom{r-1}{j}A^{-4j}\right]-1 \pmod r \\
&\equiv A^{2r-2}\left[\left(\sum_{j=0}^{r-1}\binom{r-1}{j}A^{-4j}\right)-A^{2-2r}\right] \pmod r.
\end{aligned}$$

Now, it suffices to prove that

$$\binom{r-1}{j}\equiv(-1)^j \pmod r, \tag{2}$$

because if this is true then

$$\begin{aligned}
d^{(r-1)(k-1)}-1 &\equiv A^{2r-2}\left[\left(\sum_{j=0}^{r-1}(-1)^jA^{-4j}\right)-A^{2-2r}\right] \pmod r \\
&\equiv A^{2r-2}\xi_r(A) \pmod r \\
&\equiv 0 \pmod{r,\xi_r(A)},
\end{aligned}$$

which is equivalent to

$$d^{(r-1)(k-1)} \equiv 1 \pmod{r, \xi_r(A)}$$

for all prime r and $k \in \mathbb{N}$.

To prove Equation (2), consider the identity, $\binom{r}{j} = \binom{r-1}{j} + \binom{r-1}{j-1}$. On the left-hand side, $\binom{r}{j} \equiv 0 \pmod{r}$. On the right-hand side, induction on j gives that $\binom{r-1}{j} + \binom{r-1}{j-1} \equiv \binom{r-1}{j} + (-1)^{j-1} \pmod{r}$ (Note that Equation (2) is obviously true for $j = 0$). Thus,

$$\begin{aligned} 0 &\equiv \binom{r-1}{j} + (-1)^{j-1} \pmod{r} \\ \Rightarrow \binom{r-1}{j} &\equiv (-1)^j \pmod{r}. \end{aligned}$$

This completes the proof of Lemma 1.

□

The following theorem is a relationship between the arrow polynomial of a periodic virtual link and its factor link. Murasugi (1988) proved a simple relationship between the Jones polynomial of a periodic link and the Jones polynomial of its factor link. Theorem 2 extends Murasugi's relationship to the arrow polynomial for periodic virtual links.

Theorem 2. *Let r be prime and L be a non-split virtual link that has period r^q $(q \geq 1)$. (If L has c classical crossings, then the factor link L_* has $c' = \frac{c}{r^q}$ classical crossings.) Then,*

$$\mathcal{A}[L] \equiv (\mathcal{A}[L_*])^{r^q} \pmod{r, \xi_r(A), \{\chi_{r,i}\}}$$

where

$$\xi_r(A) = \left(\sum_{j=0}^{r-1} (-1)^j A^{-4j} \right) - A^{2-2r}$$

and

$$\chi_{r,i} = K_i^r - K_{ir} \quad \left(1 \leq i \leq c'\right).$$

Proof. To begin the proof, note that it suffices to prove Theorem 2 for $q = 1$. If a link has period r^q, then it also has period r. If L is factored as a r-periodic link, the factor link L_* will still have period r^{q-1}. Then an inductive argument will apply. Therefore, in the remainder of the proof, we will assume L is r-periodic.

The normalizing factor $\left(A^3\right)^{-\omega(D)}$ in the arrow polynomial (Equation (1)) only depends on the writhe of the virtual link. By definition, the writhe of a virtual link is the total number of positive (classical) crossings minus the total number of negative (classical) crossings. Say $\tilde{L}$ has rP positive crossings and rN negative crossings. Then since $\tilde{L}$ is r-periodic, $\tilde{L}_*$ must have exactly P positive crossings and N negative crossings. Thus, $w\left(\tilde{L}\right) = rP - rN = r\left(P - N\right) = rw\left(\tilde{L}_*\right)$.

Let $\kappa = 2^c$ and $\kappa' = 2^{c'}$. Let $s_1, s_2, ..., s_\kappa$ denote all of the states of $\tilde{L}$. Likewise, let $s_{*_1}, s_{*_2}, ..., s_{*_{\kappa'}}$ denote all of the states of $\tilde{L}_*$.

Consider an arbitrary state s of $\tilde{L}$ and its corresponding factor state s_* of $\tilde{L}_*$. Denote by α and α_* the terms that s and s_* contribute to the arrow polynomial of L and L_*, respectively. From Equation (1), we see that

$$\alpha = A^{p(s)-n(s)} d^{|s|-1} \left[s\right] \tag{3}$$

$$\alpha_* = A^{p(s_*)-n(s_*)} d^{|s_*|-1} \left[s_*\right]. \tag{4}$$

We need to find a relationship between Equations (3) and (4). Consider the following two cases.

Case 1: Let s be a state which is not r-periodic. In other words, s is not fixed under ζ, i.e. $\zeta\left(s\right) \neq s$. In this case, $s, \zeta\left(s\right), \zeta^2\left(s\right), \ldots, \zeta^{r-1}\left(s\right)$ are all distinct. However, any two of these are isomorphic implying that we have r identical terms in $\mathcal{A}\left(L\right)$, and thus they vanish after reducing modulo r. Furthermore, due to this reduction, the upper bound on the number of states that will contribute to $\mathcal{A}[L]$ is κ'.

Case 2: Let s be a state that is r-periodic. In this case, $p(s) = rp(s_*)$ and $n(s) = rn(s_*)$. Therefore,

$$A^{p(s)-n(s)} = A^{r(p(s_*)-n(s_*))}.$$

However, the relationships between $|s|$ and $|s_*|$, as well as $[s]$ and $[s_*]$ are not as obvious.

First, let's consider the relationship between $|s|$ and $|s_*|$.

Let $D_1, D_2, \ldots, D_k$ be the components of s that are symmetric about $\{0\}$, i.e. $\zeta\left(D_i\right) = D_i, (1 \leq i \leq k)$.

Let $D_{1,1}, D_{1,2}, \ldots, D_{1,r}, D_{2,1}, D_{2,2}, \ldots, D_{2,r}, \ldots, D_{m,1}, D_{m,2}, \ldots, D_{m,r}$ be the components of s that are not symmetric about $\{0\}$, i.e. $\zeta(D_{i,j}) = D_{i,j+1}$ and $\zeta(D_{i,r}) = D_{i,1}$, $(1 \leq i \leq m, 1 \leq j \leq r-1)$.

Then $|s| = k + rm$ and $|s_*| = k + m$. Consider the following algebraic steps:

$$\begin{aligned}\left(d^{|s_*|-1}\right)^r &= d^{(k+m-1)r}\\ &= d^{rk+rm-r}\\ &= d^{k+rm-1+(r-1)k-r+1}\\ &= d^{k+rm-1}d^{(r-1)k-(r-1)}\\ &= d^{|s|-1}d^{(r-1)(k-1)}\end{aligned}$$

By Lemma 1, $d^{(r-1)(k-1)} \equiv 1 \pmod{r, \xi_r(A)}$ and thus,

$$\left(d^{|s_*|-1}\right)^r \equiv d^{|s|-1} \pmod{r, \xi_r(A)}.$$

Finally we must consider the relationship between $[s]$ and $[s_*]$. Recall that s has k components symmetric about $\{0\}$ and rm components not symmetric about $\{0\}$.

We need to count the number of cusp pairs in s. Consider an arbitrary component, D_i, of s symmetric about $\{0\}$. Since $\zeta(D_i) = D_i$ and ζ is a rotation of order r, the number of cusp pairs of D_i must be a multiple of r. Furthermore, recall that $\tilde{L}$ has $c'r$ classical crossings and that each classical crossing may possibly give rise to a cusp pair. Therefore the upper bound on the number of cusp pairs in D_i is $c'r$. Each D_i will contribute an extra term to $[s]$ of the form K_{ar} where a is some integer between 1 and c'.

Next consider an arbitrary component $D_{i,j}$ of s not symmetric about $\{0\}$. Recall that there are r components in s that are isomorphic to $D_{i,j}$. This means that there will be r factors of the form K_b that contribute to $[s]$. If each $D_{i,j}$ has exactly r copies in s then each $D_{i,j}$ has at most $\frac{c}{r} = c'$ resolved crossings. Thus, c' is the upper bound on the number of cusp pairs in $D_{i,j}$. For a fixed i, the set $\{D_{i,j}, 1 \leq j \leq r\}$ will contribute an extra term to $[s]$ of the form K_b^r where b is some integer between 1 and c'.

Now if we consider all the possible components of s, we may write the following expression for $[s]$:

$$[s] = \left(K_r^{u_1} K_{2r}^{u_2} \dots K_{c'r}^{u_{c'}}\right)\left(K_1^{rv_1} K_2^{rv_2} \dots K_{c'}^{rv_{c'}}\right).$$

Note that $u_1 + u_2 + \cdots + u_{c'} = k$ and $v_1 + v_2 + \cdots + v_{c'} = m$. Also, keep in mind that any u_a or v_b could be zero.

Now let's compute $[s_*]$ in a similar way. If a component D_i in s symmetric about $\{0\}$ had ar cusp pairs, then there will exist a corresponding component in s_* with a cusp pairs. Each D_i in s that contributes an extra term of the form K_{ar} in $[s]$ has a corresponding component in s_* that contributes an extra term to $[s_*]$ of the form K_a. If there were u_a components symmetric about $\{0\}$ in s then there will still be u_a components in s_*.

For a fixed i, there are exactly r components in s isomorphic to $D_{i,j}$. However, in s_* there will only be one corresponding component. If the single component in s had b cusp pairs then its corresponding component in s_* will also have b cusp pairs. Therefore, for a fixed i, the set $\{D_{i,j}, 0 \leq j \leq r\}$ in s that contributes an extra term of the form K_b^r in $[s]$ has a corresponding component in s_* that contributes an extra term to $[s_*]$ of the form K_b. Generally, if there were rv_b components not symmetric about $\{0\}$ in s then there will be v_b corresponding components in s_*.

Therefore, we calculate $[s_*]$:

$$[s_*] = \left(K_1^{u_1} K_2^{u_2} \dots K_{c'}^{u_{c'}}\right)\left(K_1^{v_1} K_2^{v_2} \dots K_{c'}^{v_{c'}}\right).$$

We need a relationship between $[s]$ and $[s_*]^r$. The implications of reducing modulo $\{\chi_{r,i}\}$ are as follows. For each i, $1 \leq i \leq c'$,

$$\begin{aligned} K_i^r - K_{ir} \equiv 0 \pmod{\chi_{r,i}} \Rightarrow K_i^r \\ \equiv K_{ir} \pmod{\chi_{r,i}} \Rightarrow K_i^{ru_i} \\ \equiv K_{ir}^{u_i} \pmod{\chi_{r,i}}. \end{aligned}$$

By applying the above logic for every i, $1 \leq i \leq c'$, we obtain that

$$K_1^{ru_1} K_2^{ru_2} \dots K_{c'}^{ru_{c'}} \equiv K_r^{u_1} K_{2r}^{u_2} \dots K_{c'r}^{u_{c'}} \pmod{\{\chi_{r,i}\}}.$$

We now obtain the desired relationship between $[s]$ and $[s_*]^r$:

$$\begin{aligned} [s_*]^r &= \left(K_1^{ru_1} K_2^{ru_2} \dots K_{c'}^{ru_{c'}}\right)\left(K_1^{rv_1} K_2^{rv_2} \dots K_{c'}^{rv_{c'}}\right) \\ &\equiv \left(K_r^{u_1} K_{2r}^{u_2} \dots K_{c'r}^{u_{c'}}\right)\left(K_1^{rv_1} K_2^{rv_2} \dots K_{c'}^{rv_{c'}}\right) \pmod{\{\chi_{r,i}\}} \\ &\equiv [s] \pmod{\{\chi_{r,i}\}}. \end{aligned}$$

Finally, we arrive at a relationship between α and α_*, (Equation (3) and Equation (4)):

$$\begin{aligned}
\alpha &= A^{p(s)-n(s)} d^{|s|-1} [s] \\
&= A^{r(p(s_*)-n(s_*))} d^{|s|-1} [s] \\
&\equiv \left(A^{p(s_*)-n(s_*)}\right)^r \left(d^{|s_*|-1}\right)^r [s] \pmod{r, \xi_r(A)} \\
&\equiv \left(A^{p(s_*)-n(s_*)}\right)^r \left(d^{|s_*|-1}\right)^r ([s_*])^r \pmod{r, \xi_r(A), \{\chi_{r,i}\}} \\
&\equiv \left(A^{p(s_*)-n(s_*)} d^{|s_*|-1} [s_*]\right)^r \pmod{r, \xi_r(A), \{\chi_{r,i}\}} \\
&\equiv \alpha_*^r \pmod{r, \xi_r(A), \{\chi_{r,i}\}}.
\end{aligned}$$

Recall the non-periodic states from Case 1 that vanished modulo r and the one-to-one correspondence between the number of periodic states of $\tilde{L}$ and the total number of states of $\tilde{L}_*$, i.e. κ'. Given that $\mathcal{A}[L]$ is a summation of all the α terms, we may relate $\mathcal{A}[L]$ with the α_* terms using the relationship above.

$$\begin{aligned}
\mathcal{A}[L] &\equiv \sum_{i=1}^{\kappa'} \alpha_i \pmod{r} \\
&\equiv \sum_{i=1}^{\kappa'} \alpha_{*i}^r \pmod{r, \xi_r(A), \{\chi_{r,i}\}}.
\end{aligned}$$

Since r is prime and we are working modulo r,

$$\begin{aligned}
\mathcal{A}[L] &\equiv \alpha_{*1}^r + \alpha_{*2}^r + \cdots + \alpha_{*\kappa'}^r \pmod{r, \xi_r(A), \{\chi_{r,i}\}} \\
&\equiv (\alpha_{*1} + \alpha_{*2} + \cdots + \alpha_{*\kappa'})^r \pmod{r, \xi_r(A), \{\chi_{r,i}\}} \\
&\equiv (\mathcal{A}[L_*])^r \pmod{r, \xi_r(A), \{\chi_{r,i}\}}.
\end{aligned}$$

The proof of Theorem 2 is complete. □

4 Application

5 Remarks

Although Theorem 2 requires a complicated reduction, it is the best possible. To see this, consider an n-component trivial link L. L has any period r and a factor link L_* which is also an n-component trivial link. Thus,

$$\mathcal{A}[L] = \mathcal{A}[L_*] = d^{n-1}$$

implying that the formula $\mathcal{A}[L] \equiv (\mathcal{A}[L_*])^r \pmod{I}$ only holds if the ideal I contains $\xi_r(A)$ (see Lemma 1). Furthermore, the extra structure that the arrow polynomial contains due to the existence of virtual crossings in a virtual link forces I to also contain $\chi_{r,i}$ due to an otherwise mathematically impossible relation to establish (since there is no other way to relate subscripts with powers).

Murasugi [ref] proved a slightly more exact result when differentiating between links with odd and even linking numbers. The reason we did not in our work is because we are proving his results using state sums, whereas extra structure in a Tait graph is needed to prove the analogous relationships (extended to periodic virtual links) to Theorem 2 in [ref]. As Tait graphs do provide the extra structure needed for Murasugi's more exact results, the proof using state sums is much more intuitive and relies on the underlying characteristics of the arrow polynomial. Note that the arrow polynomial is basically the Jones polynomial with the addition of extra structure (i.e. the extra variables defined in Section 2). Thus, if one wanted to see Murasugi's proof by way of state sums rather than Tait graphs, it suffices to read the proof here of Theorem 2 and simply ignore the pieces concerning the extra variables.

In his work, Murasugi also proved a relationship for split periodic links (Theorem 1 in [ref]). However, the connect sum is not well-defined for virtual links implying that the extension of Murasugi's result does not exist.

6 Acknowledgements

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